Solutions to / Marking scheme
Midterm Exam Cal I 22 Sept. 2014

1 a 1 pt S_n is a statement about a positive integer n

Suppose that

1 pt * S_1 is true

2 pt * $\forall n \geq 1$ S_n is true $\Longrightarrow$ S_{n+1} is true

1 pt Then S_n is true for all $n \in \mathbb{N}$

1 b * S_1 is true ?

3 pt Substitution of $n=1$ yields $1 = 1^2$ ok

* S_n is true $\Longrightarrow$ S_{n+1} is true ?

3 pt Given: $1 + 3 + \cdots + (2n-1) = n^2$

3 pt To be shown: $1 + 3 + \cdots (2n+1) = (n+1)^2$

$$\underbrace{1 + 3 + \cdots + (2n-1)}_{= n^2} + (2n+1) =$$

4 pt $$n^2 + (2n+1) = (n+1)^2 \quad \text{ok}$$

2 1 pt Using Mathematical Induction

* S_1 true ?

4 pt $f'(x) = x e^x + e^x = (x+1) e^x$ ok

* S_n true $\Longrightarrow$ S_{n+1} true ?

3 pt Given: $f^{(n)}(x) = (x+n) e^x$

3 pt To be shown: $f^{(n+1)}(x) = (x+n+1) e^x$

3 pt $f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x)$

2pt $= \frac{d}{dx}\left((x+n)\, e^{x}\right)$

$= (x+n)\, e^{x} + e^{x}$

2pt $= (x+n+1)\, e^{x}$ ok

3 1pt polar form

1pt $z = r e^{i\theta}$

2pt $z^2 = r^2 e^{2i\theta}$

4pt $1+i = \sqrt{2}\; e^{i\left(\frac{\pi}{4} + 2k\pi\right)}$ $k \in \mathbb{Z}$

2pt $z^2 = 1+i \implies \begin{cases} r^2 = \sqrt{2} \\ 2\theta = \frac{\pi}{4} + 2k\pi \end{cases}$

1pt $r = \sqrt{\sqrt{2}}$

1pt $\theta = \frac{\pi}{8} + k\pi$ $k \in \mathbb{Z}$

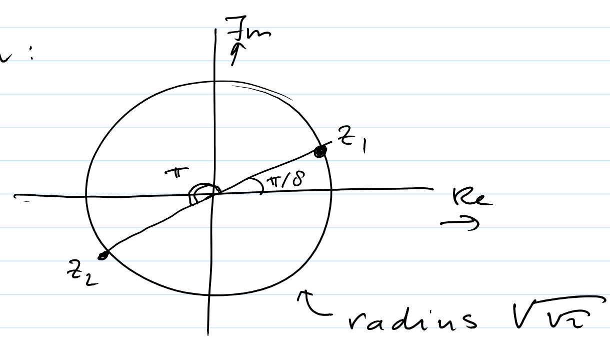
Two solutions:

1pt $z_1 = \sqrt{\sqrt{2}}\; e^{i\pi/8}$

1pt $z_2 = \sqrt{\sqrt{2}}\; e^{i9\pi/8}$

Sketch:

4 pt



4.

4pt $e^{z} = e^{-z} \implies e^{2z} = 1$

3pt $z = x+iy \implies e^{2z} = e^{2x}\, e^{2iy}$

3pt $z = x + iy \Rightarrow e^{2z} = e^{2x} e^{2iy}$

3pt $1 = 1\, e^{i(0 + 2\pi k)} \quad k \in \mathbb{Z}$

5pt $e^{2z} = 1 \Rightarrow \begin{cases} e^{2x} = 1 \Rightarrow x = 0 \\ 2y = 2\pi k \end{cases}$

Solutions

3pt $z = i\pi k$ with $k \in \mathbb{Z}$

Alternative solution method:

1pt $z = x + iy$

3pt $e^{z} = e^{x} \cdot e^{iy} = e^{x}(\cos y + i \sin y)$

$e^{-z} = e^{-x} \cdot e^{-iy} = e^{-x}(\cos(-y) + i\sin(-y))$

4pt $= e^{-x}(\cos y - i \sin y)$

4pt $e^{z} = e^{-z} \Rightarrow \begin{cases} e^{x} \cos y = e^{-x} \cos y \\ e^{x} \sin y = -e^{-x} \sin y \end{cases}$

5pt Solutions $x = 0 \quad y = \pi k \quad k \in \mathbb{Z}$

1pt Hence $z = i\pi k$ with $k \in \mathbb{Z}$

5. Definition limit $\lim_{x \to a} f(x) = L$

4pt $\forall \varepsilon > 0 \; \exists \delta > 0 : \; 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Here

4pt $\forall \varepsilon > 0 \; \exists \delta > 0 : \; 0 < |x - 3| < \delta \Rightarrow |4x - 5 - 7| < \varepsilon$

Let $\varepsilon > 0$ be given. We must find $\delta > 0$ so that

2pt if $0 < |x-3| < \delta$ then $|4x-12| < \varepsilon$

2pt if $0 < |x-3| < \delta$ then $4|x-3| < \varepsilon$

2pt if $0 < |x-3| < \delta$ then $|x-3| < \frac{\varepsilon}{4}$

Therefore, the condition of the definition is

3pt satisfied with $\delta = \frac{\varepsilon}{4}$

1pt Hence $\lim_{x \to 3} 4x-5 = 7$